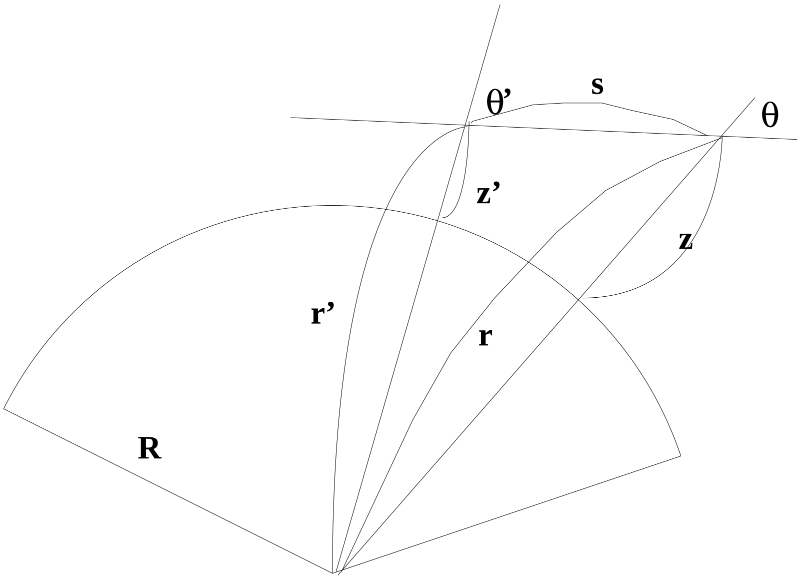

Figure 1: Geometrical relation

$$
\begin{aligned}
r\cos\theta - r'\cos\theta' &= s && (1)\\
r\sin\theta &= r'\sin\theta && (2)\\
r^2 + s^2 - 2rs\cos\theta &= r'^2 && (3)
\end{aligned}
$$

Then, we have

$$
\begin{aligned}
z' &= \sqrt{r^2+s^2-2rs\cos\theta} - R \\
&= r\sqrt{1+(\frac{s}{r})^2 - 2\frac{s}{r}\cos\theta} - R \\
&= r[1+\frac{1}{2}\{(\frac{s}{r})^2 - 2\frac{s}{r}\cos\theta\} - \frac{1}{8}\{(\frac{s}{r})^2 - 2\frac{s}{r}\cos\theta\}^2 - \frac{1}{16}8(\frac{s}{r})^3\cos^3\theta ...] - R \\
&= z - s\cos\theta + \frac{s^2}{2r}\sin^2\theta + \frac{s^3}{2r^2}\cos\theta\sin^2\theta + O(s^4) && (4)
\end{aligned}
$$

The slant thickness in the length s is

$$
\begin{aligned}
t &= \int_0^s \rho(s)ds \\
&= \int_0^s \rho(z+f(s,z))ds && (5)
\end{aligned}
$$

where

$$f(s,z) \approx -s\cos\theta + \frac{s^2}{2r}\sin^2\theta + \frac{s^3}{2r^2}\cos\theta\sin^2\theta \quad (6)$$

$$\rho(z+f(s,z)) \approx \rho(z) + \rho'(z)f(s,z) + \frac{\rho''(z)}{2}f^2(s,z) + \frac{\rho'''(z)}{6}f^3(s,z) \quad (7)$$

Taking upto s^4, the indefinit integral is

$$
\begin{aligned}
sf_1(s,z) &\equiv \int f(s,z)ds = -\frac{s^2}{2}\cos\theta + \frac{s^3}{6r}\sin^2\theta + \frac{s^4}{8r^2}\cos\theta\sin^2\theta && (8)\\
sf_2(s,z) &\equiv \int f^2(s,z)ds = \frac{s^3}{3}\cos^2\theta - \frac{s^4}{4r}\cos\theta\sin^2\theta && (9)\\
sf_3(s,z) &\equiv \int f^3(s,z)ds = -\frac{s^4}{4}\cos^3\theta && (10)
\end{aligned}
$$

Then,

$$t = s\left\{\rho(z) + \rho'(z)f_1(s,z) + \rho''(z)f_2(s,z) + \rho'''(z)f_3(s,z)\right\} \tag{11}$$

Error estimation:

$$\rho = \rho_0\exp(-z/z_0) \tag{12}$$

is an not so bad approximation for the atmosphere. Then,

$$
\begin{aligned}
\rho'(z) &= -\rho(z)/z_0\\
\rho''(z) &= \rho(z)/z_0^2 && (13)\\
\rho'''(z) &= -\rho(z)/z_0^3 && (14)
\end{aligned}
$$

$$
\begin{aligned}
t &= \rho(z)\{s + \frac{s^2}{2z_0}\cos\theta - \frac{s^3}{6rz_0}\sin^2\theta + \frac{s^3}{6z_0^2}\cos^2\theta - \frac{s^4}{8r^2z_0}\cos\theta\sin^2\theta\\
&\quad -\frac{s^4}{8rz_0^2}\cos\theta\sin^2\theta + \frac{s^4}{24z_0^3}\cos^3\theta\} && (15)\\
&= \rho(z)s\{1 + \frac{1}{2}\frac{s}{z_0}\cos\theta - \frac{1}{6}\frac{s}{r}\frac{s}{z_0}\sin^2\theta + \frac{1}{6}(\frac{s}{z_0})^2\cos^2\theta\\
&\quad -\frac{1}{8}(\frac{s}{r})^2\frac{s}{z_0}\cos\theta\sin^2\theta - \frac{1}{8}\frac{s}{r}(\frac{s}{z_0})^2\cos\theta\sin^2\theta + \frac{1}{24}(\frac{s}{z_0})^3\cos^3\theta\} && (16)\\
&= \rho(z)s\{1 + \frac{1}{2}\frac{s}{z_0}\cos\theta + \frac{1}{6}(\frac{s}{z_0})^2\cos^2\theta + \frac{1}{24}(\frac{s}{z_0})^3\cos^3\theta + ... && (17)\\
&\quad -\frac{1}{6}\frac{s}{r}\frac{s}{z_0}\sin^2\theta - \frac{1}{8}(\frac{s}{r})^2\frac{s}{z_0}\cos\theta\sin^2\theta - \frac{1}{8}\frac{s}{r}(\frac{s}{z_0})^2\cos\theta\sin^2\theta\} && (18)
\end{aligned}
$$

The first series is nothing but the one which comes from an exponential term.

For a near vertical case, we put $\cos\theta = 1$ and $\sin\theta = 0$, then

$$t = \rho(z)s\left\{1 + \frac{1}{2}\frac{s}{z_0} + \frac{1}{6}(\frac{s}{z_0})^2 + \frac{1}{24}(\frac{s}{z_0})^3\right\} \tag{19}$$

Then the relative error is order of $O((s/z_0)^4/100)$, i.e., if we take $s = 100$ m it is $\sim 10^{-9}$, since $z_0 \sim 6$ km. For near horizontal case, we put $\cos\theta = 0$ and $\sin\theta = 1$, then, we get,

$$t = \rho(z)s(1 - \frac{1}{6}\frac{s}{r}\frac{s}{z_0}) \tag{20}$$

The rellative error should be order of $O((\frac{s}{r})^2\frac{s}{z_0}) \sim 10^{-11}$ and is very small.

In practical application, if $\cos\theta \neq 0$ we may neglect the last terms in f_1 and f_2.

It is easy to get thickness corresponding to a given small s. Inversely, if a small t is given, s may be obtained by solving Eq.11:

$$s = t/\left\{\rho(z) + \rho'(z)f_1(s,z) + \rho''(z)f_2(s,z) + \rho'''(z)f_3(s,z)\right\} \tag{21}$$

This can be solved by iteration with the first $s = 0$.