1 Solution for a constant magnetic field

Equation of motion in a magnetic field.

$$\frac{d\vec{r}}{dt} = c\vec{\beta} \tag{1}$$

$$m\gamma\frac{d\vec{\beta}}{dt} = q\vec{\beta}\times\vec{B} \tag{2}$$

We assume $\vec{B}$ and $|\vec{\beta}| = \beta_0$ are constant in a short time interval. Let make the $\vec{B}$ direction to be the z-axis ($\vec{B} = (0, 0, B)$), and the x axis be directed to $\vec{\beta}\times\vec{B}$. Hereafter, all the vector components are assumed to be expressed in the (x, y, z) system unless otherwise stated (see Fig). We get

$$\beta_z = \text{const} = \beta_{z0} \tag{3}$$

$$\frac{d\beta_x}{dt} = \frac{qB}{m\gamma}\beta_y \tag{4}$$

$$\frac{d\beta_y}{dt} = -\frac{qB}{m\gamma}\beta_x \tag{5}$$

From the last two, we get

$$\frac{d^2\beta_x}{dt^2} = -\omega^2\beta_x \tag{6}$$

$$\frac{d^2\beta_y}{dt^2} = -\omega^2\beta_y \tag{7}$$

$$\omega = \frac{|q|B}{m\gamma} \tag{8}$$

We note that $\beta_\perp = \sqrt{\beta_x^2 + \beta_y^2} = \beta_{y0} = \beta_0\sin\theta$ is constant and $\beta_{x0} = 0$ (suffix 0 means at time $t = 0$). Here θ is the constant pitch angle. Then,

$$\beta_x = \pm\beta_\perp\sin(\omega t) \tag{9}$$

$$\beta_y = \beta_\perp\cos(\omega t) \tag{10}$$

where the sign in the first expression is the same as the sign of q (since $\beta_y > 0$, it follows that $d\beta_x$ should have the same sign as q).

Upon itegration, we get

$$x = \mp c\beta_\perp\frac{\cos(\omega t) - 1}{\omega} \tag{11}$$

$$y = c\beta_\perp\frac{\sin(\omega t)}{\omega} \tag{12}$$

$$z = c\beta_{z0}t \tag{13}$$

where we assumed that the particle is at the origin at time $t = 0$. The gyroradius is

$$r_G = \frac{c\beta_\perp}{\omega} = \frac{m\gamma\beta c\sin\theta}{qB} = r\sin\theta \tag{14}$$

$$r \equiv \frac{p}{qB} \tag{15}$$

where p is the momentum. We may note that if we use singed r without taking the absolute of q, the sign problem is automatically resolved ($\pm$ can be dropped, $\mp$ should read $-$. So we use the signed r

hereafter). If a particle travels a distance l in a time t,

$$
\begin{aligned}
l &= \int \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2} dt && (16) \\
\text{i.e} \quad t &= l/c\beta_0 && (17)
\end{aligned}
$$

Then,

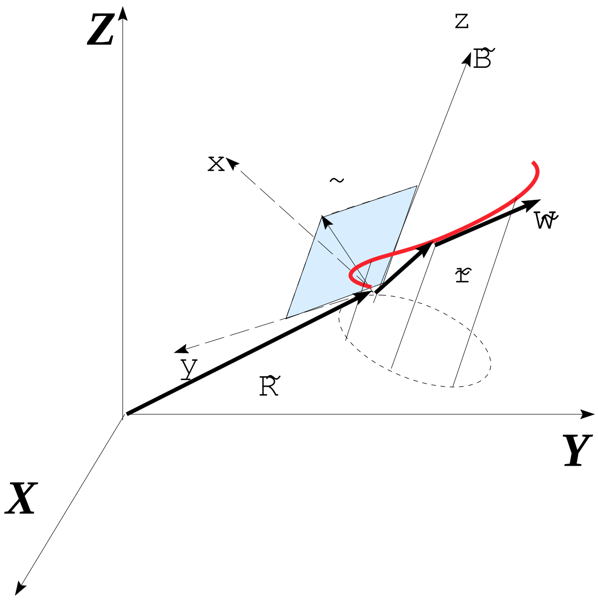
$$
\omega t = \frac{l}{r} \tag{18}
$$

The displacement vector in this time interval is $\Delta\vec{r} = (x, y, z)$ with

$$
\begin{aligned}
x &= -r \sin\theta (\cos(\frac{l}{r}) - 1) && (19) \\
y &= r \sin\theta \sin(\frac{l}{r}) && (20) \\
z &= l w_z && (21)
\end{aligned}
$$

where $w_z = w_{z0}$ is the z component of the particle direction cosines, $\vec{w} = (w_x, w_y, w_z)$, which is $\vec{\beta}/\beta_0$: That is

$$
\begin{aligned}
\vec{w} &= (\beta_\perp \sin(\omega t),\ \beta_\perp \cos(\omega t),\ \beta_{z0})/\beta_0 && (22) \\
&= (\sin\theta \sin(\frac{l}{r}),\ \sin\theta \cos(\frac{l}{r}),\ w_{z0}) && (23) \\
\sin\theta &= \sqrt{w_x^2 + w_y^2} && (24)
\end{aligned}
$$



Then, the prescription for getting the new position and the direction cosines in the original (X, Y, Z) system is:

- Form the (x, y, z) system by referring to $\vec{B}$ and $\vec{\beta}$.
- Comput $\sin\theta$ from $\vec{\beta} \cdot \vec{B} = \beta_0 B \cos\theta$
- Compute r.

- Compute $\Delta\vec{r}$ and $\vec{w}$ in the (x, y, z) sytem.
- Convert above two quantities to (X, Y, Z) system (Let them be $\Delta\vec{R}$ and $\vec{W}$).
- The new position is $\vec{R} + \Delta\vec{R}$
- The new direction cosines are $\vec{W}$.
- The only one essential value is

$$r = \frac{p}{qB} = \frac{p}{ZeB} = \frac{pc}{ZBm_ec^2}c\frac{m_e}{e} = \frac{pc}{ZB}\frac{2.998 \times 10^8}{0.511 \times 10^{-3} \cdot 1.7588 \times 10^{11}} = 3.3358\frac{pc}{ZB} \tag{25}$$

where the unit of r is in [m] with pc in [GeV] and B in [T]. For the proton of momentum 1 GeV/c in a typical $\sim 0.3 \times 10^{-4}$ [T] geomagnetic field, we get $r \sim 10^5$ [m] = 100 [km].

The conversion matrix from (x, y, z) to (X, Y, Z) is

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = T \begin{pmatrix} x \\ y \\ z \end{pmatrix} \qquad \text{or} \qquad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = T^t \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \tag{26}$$

where T^t is the transposed matrix of T which is defiend as

$$T = \begin{pmatrix} T_{xX} & T_{yX} & T_{zX} \\ T_{xY} & T_{yY} & T_{zY} \\ T_{xZ} & T_{yZ} & T_{zZ} \end{pmatrix} \tag{27}$$

T_{ij} is the j component of the direction cosines of the i-axis (in the (X, Y, Z) system). Hence,

$$\vec{T}_z = (B_x,\ B_y,\ B_z)/B \tag{28}$$

$$\vec{T}_x = \vec{w}_0 \times \vec{T}_z / \sin\theta \tag{29}$$

$$\vec{T}_y = \vec{T}_z \times \vec{T}_x \tag{30}$$

If $\sin\theta$ is zero or very small, we may take $\vec{T}_x = (1, 0, 0)$, $\vec{T}_y = (0, 1, 0)$. If $\vec{B}$ is slowly changing within the distance l, some improvement may be possible by using $\vec{B}$ at $\vec{r} = l\vec{w}/2$.

2 Runge-Kutta method

When $\vec{B}$ cannot be regarded as constant, we have to employ a numerical method for solving the differential equaitons. For this purpose it is better to rewrite them as follows. We still assume β is constant, and use $\ell = c\beta t$ as the independent variable in stead of t. Then, the basic equations become

$$\frac{d\vec{r}}{d\ell} = \vec{w} \tag{31}$$

$$\frac{d\vec{w}}{d\ell} = \frac{q}{p}\vec{w} \times \vec{B} \tag{32}$$

$$= \frac{1}{3.3358}\frac{Z}{pc}\vec{w} \times \vec{B} \tag{33}$$

where B is in [T] and pc in GeV in the last equation.